Lecture Notes for Abstract Algebra: Lecture 11

1 Group homomorphisms and normal subgroups

1.1 Group homomorphisms

Definition 1. A group homomorphism $\psi\colon (G,*) \longrightarrow (G',\star)$ is a map $\psi\colon G \longrightarrow G'$ such that

$$\psi(x * y) = \psi(x) \star \psi(y) \quad \forall\, x, y \in G.$$

Proposition 2. *Let $\psi\colon G \longrightarrow G'$ be a group homomorphism. Then:*

(1) $\psi(e_G) = e_{G'}$.

(2) For $x \in G$, $\psi(x^{-1}) = (\psi(x))^{-1}$.

(3) For any subgroup H of G and H' of G', the sets

$$\psi(H) = \{\psi(x) \,|\, x \in H\} \qquad \textit{and} \qquad \psi^{-1}(H') = \{x \in G \,|\, \psi(x) \in H'\}$$

are subgroups of G and G' respectively.

Proof. Let us denote $e = e_G$ and $e' = e'_G$.
(1) For $x = y = e$, we have $\psi(e) \star \psi(e) = \psi(e) \Rightarrow \psi(e) = e'$.
(2) For $x \in G$, we have $e' = \psi(e) = \psi(x * x^{-1}) = \psi(x) \star \psi(x^{-1}) \Rightarrow \psi(x)^{-1} = \psi(x^{-1})$.
(3) The respective neutral elements e, e' satisfy $e \in \psi^{-1}(H')$ and $e' \in \psi(H)$. Hence these two subsets are not empty. Also, for $x', y' \in \psi(H)$, we can find $x, y \in G$ such that $\psi(x) = x'$ and $\psi(y) = y'$. We have therefore

$$x' \star y'^{-1} = \psi(x) \star \psi(y)^{-1} = \psi(x * y^{-1}) \in \psi(H).$$

The statement for $\psi^{-1}(H')$ is proven similarly. □

Corollary 3. *The kernel* $\ker(\psi) = \psi^{-1}(\{e'\}) = \{x \in G \,|\, \psi(x) = e'\}$ *is a subgroup of G and the image $Im(\psi) = \psi(G) = \{y \in G' \,|\, \exists x \in G, \psi(x) = y\}$ is a subgroup of G'.*

Proposition 4. *Let $\psi\colon G \longrightarrow G'$ be a group homomorphism.*

(1) The map ψ is injective if and only if $\ker(\psi) = \{e\}$.

(2) The map ψ is surjective if and only if $Im(\psi) = G'$.

Proof. (1) Suppose that $\psi\colon G \longrightarrow G'$ is injective. The kernel always contains the neutral e. If $x \in \ker(\psi) \Rightarrow \psi(x) = \psi(e) \Rightarrow x = e$. Hence $\ker(\psi) = \{e\}$. Suppose that the kernel is trivial:

$$\ker = \{e\} \Rightarrow (\psi(x) = \psi(x') \Rightarrow \psi(x * x'^{-1}) = e' \Rightarrow x * x'^{-1} = e \Rightarrow x = x')$$

and we have injectivity of ψ. Part (2) is in general true for all maps, not just for group homomorphisms. □

Example 5. The map $\psi : (\mathrm{GL}(n,\mathbb{R}), \cdot) \longrightarrow (\mathbb{R} \setminus \{0\}, \cdot)$ given by $\psi(A) = \det(A)$ is a surjective group homomorphism with kernel $\ker(\psi) = (\mathrm{SL}(n,\mathbb{R}), \cdot)$.

Remark 6. A group isomorphism is a bijective group homomorphism. A bijective group homomorphism $\psi\colon G \longrightarrow G'$ admits and inverse $\psi'\colon G' \longrightarrow G$ which is also a group homomorphism: For all $y, y' \in G'$, we can find unique elements x, x' in G such $\psi(x) = y$ and $\psi(x') = y'$. We have

$$\psi'(y \star y') = \psi'(\psi(x) \star \psi(x')) = \psi'(\psi(x * x')) = x * x' = \psi'(y) * \psi'(y').$$

1.2 Normal subgroups

Let $\psi\colon G \longrightarrow G'$ be a group homomorphism. The subgroup $\ker(\psi) \leqslant G$ has a very peculiar property, that is, for all $x \in G$ and $h \in \ker(\psi)$,

$$\psi(xhx^{-1}) = \psi(x)\psi(h)\psi(x)^{-1} = \psi(x)e'\psi(x)^{-1} = e' \Rightarrow xhx^{-1} \in \ker(\psi).$$

Equivalently, for $x \in G$ and $h \in \ker(\psi)$, there exist $h' \in \ker(\psi)$ such that $xh = h'x$ or, in other words, we have equality of left and right cosets $xH = Hx$ for all $x \in G$.

Definition 7. A subgroup $H \leqslant G$ of a group G is said to be normal if we have equality of sets between the left and right coset of any element $x \in G$:

$$\forall x \in G \quad \Rightarrow \quad xH = Hx.$$

We use the notation $H \trianglelefteq G$.

Example 8. For an abelian group G, any subgroup $H \leqslant G$ is normal.

Remark 9. A subgroup is normal if and only if it is invariant by conjugation. This is:

$$H \trianglelefteq G \iff \varphi_x(H) = xHx^{-1} = H.$$

Example 10. Let $\psi\colon (G, *) \longrightarrow (G', \star)$ be a group homomorphism. As observed before, the kernel $\ker(\psi) \subset G$ is a normal subgroup of G. The image, in general, not need to be normal.

Example 11. Let G be a group, H a subgroup and N a normal subgroup. The subgroup $H \cap N$ is normal in H.

Example 12. Let G be a group, $H \leqslant G$ a subgroup and a normal subgroup $N \trianglelefteq G$. Then the set $HN = \{hn \,|\, h \in H,\, n \in N\}$ is a subgroup of G with $N \trianglelefteq HN$.

Proposition 13. *Let G be a group and $H \leqslant G$ a subgroup. The following are equivalent:*

(1) $H \trianglelefteq G$ (H is normal).

(2) The set of left cosets $\{xH \mid x \in G\}$ has a natural structure of group with

(a) Identity $eH = H$.

(b) $xH \cdot yH = xyH$ for any $x, y \in G$.

Proof. We define the group operation by $xH \cdot yH = xyH$. This works only if we have

$$x'H = xH\,,\ y'H = yH \Rightarrow xyH = x'y'H. \tag{1}$$

Since the group is normal

$$xyH = x(yH) = x(y'H) = x(Hy') = (xH)y' = (x'H)y' = (Hx')y' = Hx'y'.$$

On the other hand if condition 1 is satisfied, for all $x \in G$ and $h \in H$ we know that $eH = hH$ and $xH = eHxH = hHxH = hxH$. But then:

$$xH = hxH \iff x^{-1}HxH = x^{-1}hxH \iff H = x^{-1}hxH.$$

The last condition $H = x^{-1}hxH$ is equivalent to $x^{-1}hx \in H \Rightarrow xH = Hx$. □

Definition 14. Let G be a group and $N \trianglelefteq G$ a normal subgroup. The group of left cosets is called quotient group or factor group of G by H and denoted by G/H. The left coset xH of x in G/H is also refer to as the class of x in G/N and denoted by $\bar{x}$. The order of G/H is the index $[G : H]$ of H in G.

Remark 15. Let G be a group and $N \trianglelefteq G$ a normal subgroup. The quotient group G/H comes equipped with surjective group homomorphism $q\colon G \longrightarrow G/H$ given by $x \mapsto \bar{x}$. The kernel of the quotient map $\ker(q)$ is exactly N and we have an exact sequece:

$$1 \to N \xrightarrow{\alpha} G \xrightarrow{\beta} G/N \to 1,$$

where $\alpha = \mathrm{Id}$ is clearly injective, $\beta = q$ is surjective and the kernel $\ker(\beta) = N = \mathrm{Im}(\alpha)$.

Example 16. Let S_n be the symmetric group and $\{1, -1\}$, the multiplicative group of two elements. The signature map

$$\mathrm{sgn}\colon S_n \longrightarrow \{1, -1\} \quad \text{given by} \quad \sigma \mapsto \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

is a group homomorphism and the kernel $\ker(\mathrm{sgn}) = A_n$, the alternate group of even permutations. As a consequence A_n is a normal subgroup of S_n.

Practice Questions:

1. Let G be a group, H a subgroup and N a normal subgroup. Show that the subgroup $H \cap N$ is normal in H.

2. Let G be a group, $H \leqslant G$ a subgroup and a normal subgroup $N \trianglelefteq G$. Show that the set $HN = \{hn \mid h \in H,\, n \in N\}$ is a subgroup of G with $N \trianglelefteq HN$.

3. Show that a subgroup of index 2 must be normal.